Reflective and refractive systems for general two-dimensional beam transformations

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A method for designing reflective and refractive surfaces that perform general transformations on two-dimensional beams is presented. In some cases the shape of the surfaces is represented by a simple integral of an analytic expression, whereas in other specific cases it is represented as a solution of a Poisson-like equation. Finally, the possible use of noncontinuous surfaces (facets) is discussed and evaluated quantitatively. Some of the novel techniques developed are also applicable for beam transformations that are realized with diffractive systems.

1. Introduction

Optical beam transformations are often necessary for wave-front shaping[1–4] and optical processing[5–11] applications. In principle the transformations may be realized with reflective, refractive, or diffractive optics. So far, most of the research in this field has concentrated on diffractive (holographic) optical techniques[3–12] because of the relative ease in which aspheric phases can be realized with computer-generated holograms.[13] However, the needed diffractive elements must usually be used with either monochromatic or quasi-monochromatic radiation, and they may suffer from low diffraction efficiencies. Moreover, unless one uses expensive electron-beam plotters for recording the computer-generated holograms, the minimal possible grating period is typically much larger than the wavelength; the large attainable grating period leads to relatively low diffraction angles, and the resulting optical system is quite large.

For more compact, polychromatic radiation and high-efficiency beam transformations, reflecting or refracting surfaces may be exploited. The recent improvement in the technology for fabricating aspheric surfaces, especially with plastic-molding and diamond-turning techniques, offers the possibility of forming aspheric optical surfaces with high optical quality. Still, it is necessary to design the aspheric shape of the surfaces. Some designs for both reflecting[1] and refracting[2] surfaces have been reported, but they were suitable only for transformations of one-dimensional beams (or, equivalently, beams with circular symmetry). For the general two-dimensional case, no design method for the surface shape has been presented yet to our knowledge.

In this paper we report on a method for designing reflective and refractive surfaces that perform general transformations on two-dimensional beams. We begin by presenting the basic relations for the shape of the aspheric surfaces in terms of partial differential equations. Then, we discuss the existence and the intergrability of the solution of these equations, and we present an approximate solution that is always continuous. Finally, the possible use of noncontinuous surfaces (facets) is discussed and evaluated quantitatively.

2. Basic Relations

Let us consider a general transformation between an input and an output beam, both with a uniform phase. These are two types of such a transformation. The first is a transformation on the intensities

$$I_{\text{in}}(x, y) \rightarrow I_{\text{out}}(x, y), \tag{1}$$

where $I_{\text{in}}(x, y)$ represents the intensity at the input and $I_{\text{out}}(x, y)$ represents that at the output. An example for this type of transformation could be a Gaussian-to-rect transformation.[4] The second is a transformation between the coordinates of the rays:[11]

$$[x, y] \rightarrow [x'(x, y), y'(x, y)], \tag{2}$$

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where $[x, y]$ are the coordinates of the input rays and $[x', y']$ are those of the output rays. The second type of transformation is used extensively in applications of image processing (e.g., the Mellin transform[9]). These two types of transformation are connected through the following relation:

$$J(x, y, x', y') = I_{out}/I_{in}, \tag{3}$$

where $J(x, y, x', y')$ is the Jacobian of the coordinate transformation. Therefore in the paper we exploit the notation of the second type of transformation, which is more general (several different coordinate transformations may generate the same intensity transformation).

A. Reflective Surfaces

The reflective optical arrangement for performing a general coordinate transformation on a two-dimensional beam is presented in Fig. 1. It contains two mirrors, whose reflecting surfaces can be described by $z(x, y)$ and $z'(x', y')$. As shown, a typical input ray incident in the z direction with coordinates $[x, y]$ reflects from the two surfaces and emerges again in the z direction but with new coordinates $[x'(x, y), y'(x, y)]$. Since we require that the phases across the input and output beams be uniform, the optical path of all the rays between two reference planes, say $z = 0$ and $z = a$, must be constant. This requirement may be expressed as

$$a + z' - z + r = c, \tag{4}$$

where

$$r = [(x - x')^2 + (y - y')^2 + (z - z')^2]^{1/2}$$

and c is a constant.

To find the shape of the two reflective surfaces, we exploit the analytic ray-tracing approach.[14] The propagation vectors (normalized) of the input ray $\mathbf{K}_{in}$ and the intermediate ray $\mathbf{K}_{int}$ (the ray reflected from the first surface) are, respectively,

$$\mathbf{K}_{in} = (0, 0, 1), \tag{5}$$

$$\mathbf{K}_{int} = \left(\frac{x' - x}{r}, \frac{y' - y}{r}, \frac{z' - z}{r}\right). \tag{6}$$

According to Snell's reflection law, the vector normal to the surface at the point (x, y, z) is

$$\mathbf{K}_{m} = \mathbf{K}_{in} - \mathbf{K}_{int} = \left(\frac{x' - x}{r}, \frac{y' - y}{r}, \frac{z' - z}{r} - 1\right). \tag{7}$$

In order to determine the shape of the surface, we must find $(\partial z)/(\partial x)$ and $(\partial z)/(\partial y)$. To find $(\partial z)/(\partial x)$, we look for a vector in the xz plane $(K_{\|x}, 0, K_{\|z})$ that is perpendicular to $\mathbf{K}_m$. This requires that the scalar product be zero:

$$(K_{m_x}, K_{m_y}, K_{m_z}) \cdot (K_{\|x}, 0, K_{\|z}) = 0. \tag{8}$$

Combining Eqs. (7) and (8), we get

$$\frac{\partial z}{\partial x} = \frac{K_{\|z}}{K_{\|x}} = -\frac{K_{m_x}}{K_{m_z}} = \frac{x' - x}{z' - z + r}, \tag{9}$$

and in a similar way,

$$\frac{\partial z}{\partial y} = \frac{y' - y}{z' - z + r}. \tag{10}$$

Finally, incorporating Eq. (4) yields

$$\frac{\partial z}{\partial x} = \frac{x' - x}{c'}, \quad \frac{\partial z}{\partial y} = \frac{y' - y}{c'}, \tag{11}$$

where $c' = c - a$ is a constant. Note that Eq. (11) is identical to the surface equation of the one-dimensional case, given in Ref. 1, with only a slight difference in the constant c'. The shape of the second reflective surface is readily found from Eq. (11) and the requirement that the input and output rays are parallel; i.e., the slopes at the corresponding points on the two surfaces are the same:

$$\frac{\partial z'}{\partial x'} = \frac{\partial z}{\partial x}, \quad \frac{\partial z'}{\partial y'} = \frac{\partial z}{\partial y}. \tag{12}$$

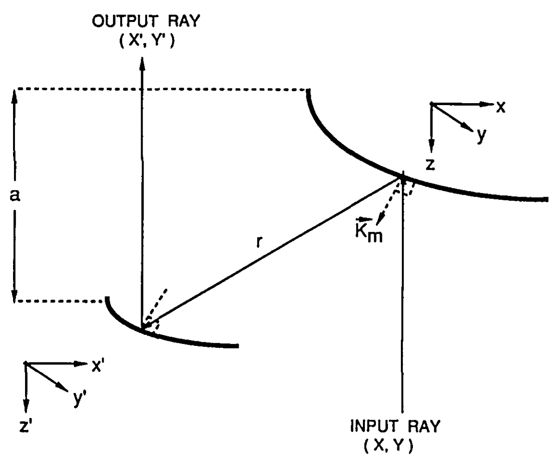

Fig. 1. Reflective optical arrangement for performing a general coordinate transformation $(x, y) \rightarrow (x', y')$ on a two-dimensional beam.

B. Refractive Surfaces

The refractive optical arrangement for performing a general coordinate transformation on a two-dimensional beam is presented in Fig. 2. It consists of a bulk substrate with an index of refraction $n > 1$, whose front and back refractive surfaces are depicted by $z(x, y)$ and $z'(x, y)$. To find the shapes of these two surfaces, we follow a similar procedure to that for the reflecting surfaces. Here, however, the requirement for uniform phases in the input and the output beams

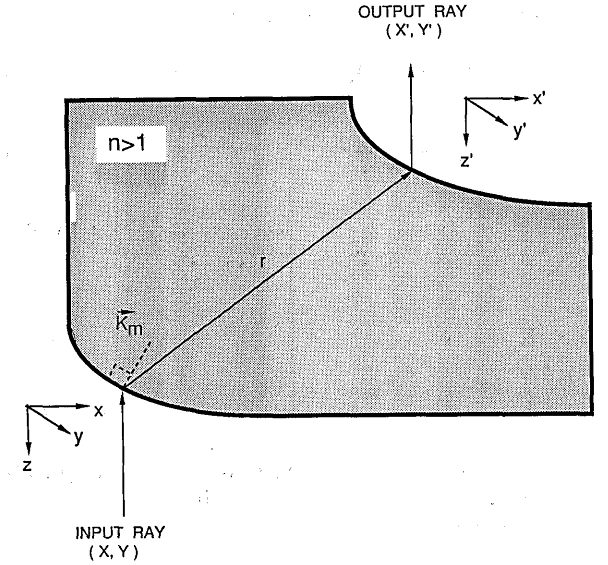

Fig. 2. Refractive optical arrangement for performing a general coordinate transformation $(x, y) \rightarrow (x', y')$ on a two-dimensional beam.

has to include the refractive index and is expressed as

$$a + z' - z + nr = c, \tag{13}$$

where $n = 1$ is assumed outside the bulk.

According to Snell's refraction law, the vector normal to the first refractive surface at the point (x, y, z) is

$$\mathbf{K}_m = \mathbf{K}_{\text{in}} - \frac{1}{n}\mathbf{K}_{\text{int}} = \left(\frac{x' - x}{r}, \frac{y' - y}{r}, \frac{z' - z}{r} - \frac{1}{n}\right), \tag{14}$$

where $\mathbf{K}_{\text{in}}$ and $\mathbf{K}_{\text{int}}$ are the same as in the reflective case [Eqs. (5) and (6), respectively]. Finally, the surface derivatives are found by the same procedure as for Eqs. (7)–(9) to be

$$\frac{\partial z}{\partial x} = -\frac{K_{m_x}}{K_{m_z}} = \frac{x' - x}{z' - z + r/n}, \tag{15}$$

$$\frac{\partial z}{\partial y} = \frac{y' - y}{z' - z + r/n}. \tag{16}$$

There is a significant difference between the reflective and the refractive case. In the reflective case the denominator on the right side of Eqs. (9) and (10) is equal to the optical path and, from Eq. (4), is simply a constant, so the differential equations reduce to simple equations. On the other hand, in the refractive case the denominator of the right side of Eqs. (15) and (16) is not a constant. Nevertheless, by incorporation of the constant optical path requirement [Eq. (13)] and after some algebraic manipulation, Eqs. (15) and (16) also reduce to

$$\frac{\partial z}{\partial x} = \frac{n(x' - x)}{\{c^2 - (n^2 - 1)[(x' - x)^2 + (y' - y)^2]\}^{1/2}}, \tag{17}$$

$$\frac{\partial z}{\partial y} = \frac{n(y' - y)}{\{c^2 - (n^2 - 1)[(x' - x)^2 + (y' - y)^2]\}^{1/2}}. \tag{18}$$

Note that in the one-dimensional case $(y' = y)$, Eq. (17) reduces to

$$\frac{\partial z}{\partial x} = \frac{n}{\{[c/(x' - x)]^2 - n^2 + 1\}^{1/2}} \tag{19}$$

which is identical to the one-dimensional slope equation that is given in Ref. 2. Finally, the shape of the second refractive surface is found, as in the reflective case, by Eq. (12).

3. Integrability

The realization of general coordinate transformations with diffractive systems has been extensively investigated.[3–12] Fortunately Eq. (11) for the shape of the reflective surfaces is exactly identical to the equation for the phase functions for the diffractive elements in paraxial beam-transformation systems[5] [although Eq. (11) itself is certainly not paraxial]. Therefore it is possible to exploit here some results that were derived earlier for diffractive systems (later in the paper derive some new results concerning noncontinuous surfaces that can be exploited for beam transformations with diffractive elements). The equations for the surfaces shape in the refractive case are more complicated and cannot be adapted directly. Nevertheless, the same basic approaches may be applied also to them, with some minor modifications.

To begin, we rewrite Eq. (11) in vectorial notation as

$$\nabla z = \mathbf{K}_d, \tag{20}$$

where

$$\mathbf{K}_d = \left(\frac{x' - x}{c'}, \frac{y' - y}{c'}\right)$$

is the desired derivative vector and $\nabla = (\partial/\partial x, \partial/\partial y)$ is the gradient operator. A unique solution to Eq. (11) exists only for transformations in which the derivative vector is a conserving vector,[14] i.e., if

$$\nabla \times \mathbf{K}_d = 0, \tag{21}$$

where $\nabla\times$ is the curl operator. For the reflective system, Eq. (21) reduces to

$$\frac{\partial x'}{\partial y} = \frac{\partial y'}{\partial x}. \tag{22}$$

This condition ensures that integration of Eq. (11) along any path in the xy plane yields an identical result. For example, using the $(0, 0) \to (x, 0) \to (x, y)$ path leads to the expression

$$z(x,y) = \frac{1}{c'}\left[\int_0^x x'(t, y=0)\mathrm{d}t + \int_0^y y'(x,t)\mathrm{d}t - (x^2+y^2)/2\right] \tag{23}$$

and to a similar expression for $z'(x', y')$.

A common example in which such a procedure is applicable is the $\ln r - \theta$ coordinate transformation[9] (which is used for scale and rotation invariance in optical correlators), given as

$$x'(x, y) = \ln(x^2 + y^2)^{1/2},$$
$$y'(x, y) = -\tan^{-1}(y/x). \tag{24}$$

Incorporating Eq. (24) into Eq. (23) and integrating yields the desired surface shape as

$$z(x, y) = [x \ln(x^2 + y^2)^{1/2} - y \tan^{-1}(y/x) - x - (x^2 + y^2)/2]/c', \tag{25}$$

and, it yields a similar expression for $z'(x', y')$.

For transformations in which the requirement of Eq. (22) is not fulfilled there is no unique solution for the reflective surface shape. There are several approaches to overcome this difficulty. First, it has been shown[12] that any nonintegrable transformation $(x, y) \to (x', y')$ may be divided into two stages $(x, y) \to (x'', y'') \to (x', y')$, in which each stage is now integrable, i.e., fulfills the requirement of Eq. (22). Note, however, that this solution involves a doubling of the optical system, and therefore it requires four surfaces instead of two.

A second approach for nonintegrable transformations, again adopted from diffractive systems,[11] is to find an optimal surface $z_{\mathrm{op}}(x, y)$ in the sense that its partial derivatives are closest to the desired ones of Eq. (11). Specifically, if the l_2 metric in the Hilbert space of two-dimensional functions is exploited, one may find the optimal surface by solving a Poisson-like equation[11]:

$$\nabla^2 z_{\mathrm{op}} = \frac{1}{c'}\left(\frac{\partial x'}{\partial x} + \frac{\partial y'}{\partial y} - 2\right), \tag{26}$$

where $\nabla^2 = (\partial^2/\partial x^2) + (\partial^2/\partial y^2)$ is the Laplacian operator. This equation can be solved numerically by standard computer programs when (a) the proper boundary conditions are $(\partial z)/(\partial \mathbf{n}) = [\mathbf{n}(x' - x, y' - y)]c'$ on the boundary of the surface, with $\mathbf{n}$ as a unit vector normal to the boundary in the xy plane, and (b) when the constant of integration can be arbitrarily chosen.

Although the procedure that leads to Eq. (26) ensures the best continuous solution for the surface function $z_{\mathrm{op}}(x, y)$, it can still yield considerable distortions in cases of a highly nonconserving derivative vector. In such cases it may be advantageous to permit some discontinuities in the surface shape or derivatives. A quantitative description of these discontinuities is presented in Section 4.

4. Noncontinuous Surfaces

The simplest way to include discontinuities in the surface shape is to divide it into small planar facets, whose angles are chosen to satisfy Eq. (11) for a certain point within each facet. This is known as the integration mirror, or multifacet, approach.[10] Unfortunately, in general, Eq. (11) can be adequately fulfilled only for a single point within each planar facet, whereas for the other points, severe aberrations exist. In order to minimize these aberrations, one must reduce the size of each facet. This reduction in size of facets leads to a large number of facets and thereby to high diffraction effects from the many discontinuities between the facets, so the overall optical performance remains relatively poor. Specifically, by adapting the results from diffractive beam-transformation systems,[10,11] one can readily show that the optimal facet size is approximately $\sqrt{\lambda c}$ and that the total distortion (aberration plus diffraction) is approximately $2\sqrt{\lambda c}$, where λ is the wavelength of the light and c is the optical path.

In order to improve the optical performance, one can exploit nonplanar facets, the shape of each facet is obtained by use of the optimization procedure of Section 3 [Eq. (26)]. Here, the optimized shape of the facet is much closer to the desired shape described by Eq. (11), so the number of facets and hence the diffraction from the discontinuities between them may be reduced considerably.

For a quantitative analysis of the optical performance and for determination of the optimal facet size, we limit ourselves again to the simpler case of reflective surfaces. We begin by characterizing the amount of nonintegrability of the transformation with a dimensionless constant R, defined as

$$R = \text{maximum}\{\nabla \cdot c'\mathbf{K}_d\} = \text{maximum}\left\{\frac{\partial x'}{\partial y} - \frac{\partial y'}{\partial x}\right\}. \tag{27}$$

For an integrable transformation, for which Eq. (22) is fulfilled, the amount of nonintegrability is indeed $R = 0$. However, for nonintegrable transformation ($R \neq 0$), the integration of Eq. (11) along different paths in the xy plane yields different surface shapes $z(x, y)$, instead of a unique one as for integrable transformations. The error in the reflecting surface shape δz that is caused by the nonintegrability may thus be estimated by these differences in shape to yield

$$\delta z \approx \oint \mathbf{K}_d \mathrm{d}\mathbf{l}, \tag{28}$$

where $\oint$ represents a closed orbital integral within the facet. By exploiting Green's theorem we can simplify relation (28) to

$$\delta z \approx \iint (\nabla \cdot \mathbf{K}_d) \mathrm{d}\mathbf{s} < \iint (R/c') \mathrm{d}\mathbf{s} \approx \frac{R(X_f)^2}{c'}, \quad (29)$$

where the double integral is on the area contained within the closed orbit, approximated by the total area of the facet $(X_f)^2$, with X_f a typical size of the facet.

Now the angular aberrations of the facet shape may be estimated by

$$\delta K_d \approx \frac{\delta z}{X_f} \approx \frac{RX_f}{c'}, \quad (30)$$

and the corresponding lateral aberrations in the output plane are

$$W_{ab} \approx c' \delta K \approx RX_f. \quad (31)$$

Relation (31) indicates that the approximated lateral aberrations are proportional to the amount of nonintegrability R and the facet size X_f. For simplicity, no distinction between c and c' was made in the derivation; this may result in an error of approximately a factor of 2 in the aberrations. On the other hand, the diffraction effects in the output are inversely proportional to the facet size and may be approximated by

$$W_{diff} \approx \frac{\lambda c}{X_f}, \quad (32)$$

where λ is the wavelength of the light. The combined effect of the aberrations and diffraction may be crudely approximated by their sum as

$$W_{tot} \approx W_{diff} + W_{ab}. \quad (33)$$

To estimate the optimal facet size $X_{f,opt}$ for which W_{tot} is minimal, we take the derivative of W_{tot} with respect to X_f and set it to zero, which yields

$$X_{f,opt} \approx \left(\frac{\lambda c}{R}\right)^{1/2}, \quad (34)$$

where some constants, of the order of 1, were omitted all along the derivation. Relation (34) indicates that the optimal facet size increases for transformations with a higher amount of integrability (smaller R), as may indeed be expected. For the optimal facet size the total distortion of the transformation (aberration plus diffraction) is

$$W_{tot} \approx 2(\lambda c R)^{1/2}. \quad (35)$$

By comparison of the results of relations (34) and (35) to those from the earlier multifacet approach,[10] two main conclusions emerge:

(1) The size of the optimal facet $X_{f,opt}$ is $1/\sqrt{R}$ times larger here than that for the multifacet approach. The area of the optimal facet $X_{f,opt}^2$ is therefore $1/R$ times larger than that for the multifacet approach, and the number of facets is $1/R$ times smaller.

(2) The total distortion in each spatial direction is $1/\sqrt{R}$ times smaller here than for the multifacet approach, so the two-dimensional space–bandwidth product is $1/R$ times larger. Consequently a substantial improvement is obtained for transformations with a low amount of nonintegrability ($R \ll 1$).

5. Concluding Remarks

A method for designing reflective and refractive surfaces that perform general transformations on two-dimensional beams has been presented. The basic relations for the shape of the needed aspheric surfaces were derived in terms of partial differential equations with an explicit form. The explicit form of these equations may be attributed to the restriction that the input and the output beams are collimated (constant optical path for all rays). For noncollimated input or output beams the partial differential equations for the surface shapes would have an implicit form and are much more difficult to solve. Even for the explicit form there are cases in which the equations for the surface shape do not have a continuous solution. For these cases we derived a formalism (adapted from diffractive optics) to obtain an optimal shape that would minimize the optical aberrations as a solution of a Poisson-like equation. Finally, we analyzed the optical performances of noncontinuous surfaces (facets), and we found the optimal facet size as a function of the amount of nonintegrability of the beam transformation.

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